SYMMETRY REDUCTION OF BROWNIAN MOTION
AND
QUANTUM CALOGERO-MOSER MODELS

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ABSTRACT. Let Q be a Riemannian G-manifold. This paper is concerned with the symmetry reduction of Brownian motion in Q and ramifications thereof in a Hamiltonian context. Specializing to the case of polar actions we discuss various versions of the stochastic Hamilton-Jacobi equation associated to the symmetry reduction of Brownian motion and observe some similarities to the Schrödinger equation of the quantum free particle reduction as described by Feher and Pusztai [11]. As an application we use this reduction scheme to derive examples of quantum Calogero-Moser systems from a stochastic setting.

CONTENTS

INTRODUCTION

Let (Q, μ) be a Riemannian manifold and G a Lie group acting properly and by isometries on Q. It is well-known that one can describe geodesic motion in Q in terms of a G-invariant Hamiltonian system $(T^*Q, \Omega^Q, \mathcal{H})$ where Ω^Q is the canonical symplectic form on T^*Q and $\mathcal{H}$ is the kinetic energy Hamiltonian. Because of the G-invariance this system may be reduced by means of the standard momentum map $J_G : T^*Q \to \mathfrak{g}^*$ at a given coadjoint orbit level $\mathcal{O} \subset \mathfrak{g}^*$ to yield the reduced Hamiltonian system $(J_G^{-1}(\mathcal{O})/G, \Omega_{\mathrm{red}}, \mathcal{H}_{\mathrm{red}})$. The reduced system may be a stratified system in the sense of [37, 34]. Solution curves in $J_G^{-1}(\mathcal{O})/G$ can be projected to curves in Q/G where they describe the evolution of a mechanical system subject to a possibly spin-dependent potential. Hence the reduced system is -in this case- more complicated (and more interesting) than the original upstairs system. Similar remarks apply to the quantum version of this procedure: Quantum Hamiltonian reduction of the free particle system will induce a new (and generally non-free) quantum system on the reduced space Q/G. See [21, 9, 11, 15].

This paper is concerned with the stochastic version of this scheme. The idea is that the stochastic analogue of a free system is Brownian motion. To further the analogy we consider the Hamiltonian description (of [25]) of Brownian motion in Q and discuss its reduction with respect to the symmetry group G. While this construction is very close to the classical one in its approach we can also use the stochastic Hamilton-Jacobi equation to obtain the Schrödinger operator of the reduced quantum free particle system as described by [11]. There are several versions of the stochastic Hamilton-Jacobi equation. We comment first on the one of [12, 30] and then use the formulation of [27]. In order to obtain a reduced system that more accurately reproduces the reduced quantum Hamiltonian operator we also use a "time-forward"-analogue of the stochastic action of [27].

Classical and quantum Calogero-Moser systems can be constructed as projections of the free system when the configuration space is a semi-simple Lie group or Lie algebra and the symmetry group is the group itself acting by conjugation or adjoint action. See [21, 9, 11, 15]. We make a certain choice in this regard leading to rational versions of these systems. Thus we can employ the above outlined procedure to obtain a stochastic version of Calogero-Moser models and pass, via the Hamilton-Jacobi equation associated to the "time-forward" stochastic action, to a stochastic representation of the quantum Calogero-Moser Schrödinger equation as well as its stationary solution wave functions.

Description of contents. Section 1 starts by providing some general background on diffusions on manifolds. Then we state a result of [19] which allows for for symmetry reductions of diffusions defined in terms of a Stratonovich operator which is equivariant in a certain sense with respect to a group action.

In Section 2 we consider the general problem of Hamiltonian construction of Brownian motion in (Q, μ) as well as some reduction issues. This is mostly independent from Section 3 but interesting in its own right. The construction of Brownian motion in (Q, μ) via the orthogonal fame bundle P is well understood, see [20]: The idea is that one rolls the manifold Q along Brownian paths in $\mathbb{R}^n$ ($n = \dim Q$) without slipping or twisting (*rubber rolling*). The rolling is defined in terms of local isometries between Q and $\mathbb{R}^n$ thus involving the orthogonal frame bundle. This gives rise to a Stratonovich equation on P and its solution diffusion process Γ^P projects to a diffusion Γ^Q in Q which can be shown to coincide with Brownian motion. Now the Hamiltonian version of [25] of this construction amounts to lifting the Stratonovich equation on P to a Stratonovich equation on T^*P which is defined in terms of Hamiltonian vector fields on T^*P associated to appropriate momentum functions. Thus we obtain a diffusion Γ^{T^*P} which projects to $\Gamma^P = \tau^P \circ \Gamma^{T^*P}$ via the foot point projection $\tau^P : T^*P \to P$ and thus ultimately to Brownian motion Γ^Q in Q.

Now, in line with general experience in Hamiltonian mechanics, one would expect that there should also be a way to induce a diffusion Γ^{T^*Q} in $T^*Q = T^*P /\!/_0 K$ from Γ^{T^*P} such that $\Gamma^Q = \tau^Q \circ \Gamma^{T^*Q}$ where $\tau^Q : T^*Q \to Q$. Here $T^*P /\!/_0 K = J_K^{-1}(0)/K$ denotes the symplectic reduction of T^*P at the 0-level set with respect to the standard momentum map J_K of the principal K-action on P cotangent lifted to T^*Q. This is certainly possible if the manifold Q is parallelizable. However, in general Γ^{T^*P} does not preserve level sets of J_K. To overcome this deficency we redo the Hamiltonian construction of Γ^{T^*P} from a *non-holonomic* point of view. Thus we obtain a different diffusion $\Gamma^{\mathcal{C}}$ which remains on $J_K^{-1}(0)$, projects to a diffusion Γ^{T^*Q} on $T^*Q = J_K^{-1}(0)/K$, and retains the basic feature $\Gamma^P = \tau^P \circ \Gamma^{\mathcal{C}}$ whence also $\Gamma^Q = \tau^Q \circ \Gamma^{T^*Q}$. It is maybe not surprising that this works since the very idea of constructing Brownian motion via rubber rolling is non-holonomic in its nature.

We also make some comments on how these Hamiltonian and non-holonomic (or almost Hamiltonian) constructions behave in the presence of a symmetry group G acting properly and by isometries on Q.

Section 3 is the main part of the paper. We assume that the G-action on (Q, μ) is actually *hyper-polar* which means that there exists an embedded submanifold $M \subset Q$ which meets all G-orbits and does so orthoganally and that M is locally isometrically diffeomorphic to Euclidean space $\mathbb{R}^l$. Moreover, it is assumed that the G-action is of single orbit type. This ensures Q to be paralellizable. Then we consider two different types of Hamiltonian constructions of Brownian motion in Q. Since the configuration space is parallelizable one can give a Hamiltonian construction of Brownian motion in Q by choosing a global orthonormal basis and corresponding momentum fuctions on T^*Q. The two choices which we consider are firstly that of a constant frame (assuming it exists) and secondly that of a G-invariant frame adapted to the decomposition into horizontal and vertical space (such a frame can be constructed under the standing assumptions). Thus we get two different Hamiltonian diffusions in T^*Q both of which project to Brownian motion in Q.

Then we consider the symmetry reductions of these diffusions to Q/G, $(T^*Q)/G$ and, where possible, to $J_G^{-1}(\mathcal{O})/G \subset (T^*Q)/G$ where $J_G : T^*Q \to \mathfrak{g}^*$ is the standard momentum map of the cotangent lifted G-action on T^*Q and $\mathcal{O} \subset \mathfrak{g}^*$ is a coadjoint orbit.

At the Q/G-level we may consider the stochastic Hamilton-Jacobi equation of Guerra-Morato [12] and Nelson [30]. If the orbit $\mathcal{O}$ is such that $J_G^{-1}(\mathcal{O})/G$ is diffeomorphic to a cotangent bundle (this happens e.g. with $\mathcal{O} = 0$), then we can also invoke the stochastic Hamilton-Jacobi equation of Lazaro-Cami and Ortega [27] associated to the projected stochastic action $\widetilde{S}$ with respect to a choice of a (regular) Lagrange L_f submanifold

in $J_G^{-1}(\mathcal{O})/G$. With

$$\psi(t,x) := \delta^{-\frac{1}{2}}(x)E[\exp(-\widetilde{S}_t^x)],$$

where δ is a function on B depending on the inertia tensor $\mathbb{I}$ associated to the metric μ we thus find the following diffusion equation:

$$(3.33) \qquad \frac{\partial}{\partial t}\psi = \left(\tfrac{1}{2}\Delta - \tfrac{1}{2}\delta^{\frac{1}{2}}\Delta\delta^{-\frac{1}{2}} + \tfrac{1}{2}\langle\lambda, \mathbb{I}_x^{-1}(\lambda)\rangle\right)\psi.$$

This equation is analoguous to the Schrödinger equation of quantum free partice reduction (with respect to polar actions) described by Feher and Pusztai [11, Thm. 4.5].

We treat the case $\mathcal{O} = 0$ separately in Section 3.E and show that stationary solutions to (3.33) are linked to Lagrange submanifolds L_f associated to eigenfunctions f of the radial part of the Laplace-Beltrami operator on (Q, μ). To obtain a diffusion equation which more accurately reproduces the Schrödinger operator of the reduced quantum free particle system we also introduce a "time-forward" formulation of the projected stochastic action used by [27] to obtain their version of the stochastic Hamilton-Jacobi equation.

Finally in Section 4 we apply the results of Section 3 to obtain stationary solutions to (rational) quantum Calogero-Moser models associated to semi-simple Lie algebras. This reproduces (and makes use of) some of the formulas obtained by Olshanetsky and Perelomov [32, 33].

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1. SOME STOCHASTIC GEOMETRY

This section begins with a review of some necessary definitions and results that are all contained in the books [20, 8]. Then we state Theorem 1.3 which provides one of the approaches to be used in subsequent symmetry reduction schemes.

1.A. **Diffusions on manifolds.** A diffusion is a continuous stochastic process which has the strong Markov property. This is a concept which can be formulated in any (decent) topological space.

Let X be a locally compact topological space with one-point compactification $\dot{X} = X \cup \{\infty\}$ and furnish $\dot{X}$ with its Borel σ-algebra $\mathcal{B}(X)$. Define $W(X)$ to be the set of all maps $w : [0,\infty) \to \dot{X}$ such that there is a $\zeta(w) \in [0,\infty]$ satisfying

(1) $w(t) \in X$ for all $t \in [0, \zeta(w))$ and $w : [0, \zeta(w)) \to X$ is continuous;
(2) $w(t) = \infty$ for all $t \geq \zeta(w)$.

Now $W(X)$ is equipped with the σ-algebra $\mathcal{B}(W(X))$ generated by all Borel cylinder sets in $W(X)$. This σ-algebra has a natural filtration given by the family of $(\mathcal{B}_t(W(X)))_{t\geq 0}$ which are the σ-algebras generated by Borel cylinder sets up to time t.

A family of probabilities $(P_x)_{x\in\dot{X}}$ on $(W(X), \mathcal{B}(W(X)))$ is said to be a *system of diffusion measures* on $(W(X), \mathcal{B}(W(X)), \mathcal{B}_t(W(X)))$ if it has the strong Markov property, for the definition of which we refer to [20, Section IV.5].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\Gamma : \Omega \times \mathbb{R}_+ \to \dot{X}$ a map. Define $\check{\Gamma} : \omega \mapsto (t \mapsto \Gamma_t(\omega))$. Then Γ is said to be a (continuous) stochastic process in X if $\check{\Gamma} : (\Omega, \mathcal{F}) \to (W(X), \mathcal{B}(W(X)))$ is a random variable.[1] The law of Γ is by definition the push-forward probability $\check{\Gamma}_*P$ on $(W(X), \mathcal{B}(W(X)))$, i.e., $\check{\Gamma}_*P(S) = P(\check{\Gamma}^{-1}(S))$ for all $S \in \mathcal{B}(W(X))$.

The process Γ is a *diffusion* in X if there is a system of diffusion measures $(P_x)_{x\in\dot{X}}$ such that $\check{\Gamma}_*P = P_\mu$ as probability laws on $(W(X), \mathcal{B}(W(X)))$; here

$$P_\mu(S) = \int_{\dot{X}} P_x(S)\mu(dx) \quad \text{for all } S \in \mathcal{B}(W(X))$$

and $\mu = (\Gamma_0)_*P : \mathcal{B}(X) \to [0,1]$ is the initial distribution of Γ. In practice P_x will be obtained by push-forward of P with respect to the stochastic process Γ^x which is conditioned such that $\Gamma_0^x = x$ a.s.

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[1] We consider only continuous processes.

Diffusions via Stratonovich equations. Let now $X = Q$ be a manifold. If N is another manifold then a *Stratonovich operator* $\mathcal{S}$ from TN to TQ is a section of $T^*N \otimes TQ \to N \times Q$. Equivalently we can view $\mathcal{S}$ as a smooth map $\mathcal{S} : Q \times TN \to TQ$ which is linear in the fibers and sits over the identity on Q. Let $X_0, X_1, \ldots, X_k$ be vector fields on Q and define the associated Stratonovich operator $\mathcal{S}$ from $T\mathbb{R}^{k+1}$ to TQ by

$$\mathcal{S} : Q \times T\mathbb{R}^{k+1} \longrightarrow TQ, \; (x, w, w') \longmapsto \sum_{i=0}^{k} X_i(x)\langle e_i, w'\rangle$$

where e_i denotes the standard basis in $\mathbb{R}^{k+1}$. We remark that the number k is not related to the dimension of Q. Assume $(\mathcal{F}_t)$ is an increasing filtration of $\mathcal{F}$ which is right-continuous. It is then a reference family in the sense of [20, p. 20]. Consider the process $Y : \Omega \times \mathbb{R}_+ \to \mathbb{R}^{k+1}$, $(t, \omega) \mapsto (t, W_t(\omega))$, where W denotes a continuous version of $(\mathcal{F}_t)$-adapted Brownian motion in $\mathbb{R}^k$. We will be concerned with Stratonovich equations of the form

$$(1.1) \qquad\qquad\qquad\qquad\qquad \delta\Gamma = \mathcal{S}(Y, \Gamma)\delta Y.$$

A continuous $(\mathcal{F}_t)$-adapted process $\Gamma : \Omega \times \mathbb{R}_+ \to Q$ is called a solution to (1.1) if there is a continuous version $W = (W^i)$ of $(\mathcal{F}_t)$-adapted Brownian motion in $\mathbb{R}^k$ such that, in the Stratonovich sense,

$$(1.2) \qquad\qquad f(\Gamma_t) - f(\Gamma_0) = \int_0^t (X_0 f)(\Gamma_s)ds + \sum_{i=1}^{k} \int_0^t (X_i f)(\Gamma_s)\delta W_s^i$$

for all smooth functions f of compact support on Q.

Suppose Γ is a solution to (1.1) such that $\Gamma_0 = x$ a.s. and Γ satisfies (1.2) with respect to a version W. Then we will write $\Gamma = \Gamma^{x,W}$ to remember these data. The following is an account of Theorems V.1.1 and V.1.2 in [20]

Theorem 1.1. *Let the assumptions be as above and consider Equation* (1.1).

(1) *For each initial condition, $\Gamma_0 = x$ a.s., and continuous $(\mathcal{F}_t)$-adapted Brownian motion W, a solution $\Gamma^{x,W}$ exists and is unique up to explosion time.*
(2) *Let $P_x := \check{\Gamma}_*^{x,W} P$. Then P_x is independent of W and (P_x) is a system of diffusion measures generated by the second order differential operator*

$$(1.3) \qquad\qquad\qquad\qquad\qquad A = X_0 + \tfrac{1}{2}\sum_{i=1}^{k} X_i X_i.$$

which acts on the space of smooth functions with compact support $C^\infty(Q)_0$.

In general, if Γ is a diffusion in Q such that the associated system of diffusion measures is unique and is generated by a second order differential operator A, then A is also called the generator of Γ and Γ is said to be an A-diffusion. This does not require Q to be a manifold; if Q is a topological space then a generator is a linear operator A on the Banach space of continuous functions $C(\check{Q})$ with domain of definition $\mathcal{D}(A)$. See [20, Section IV.5].

Assume that Q is endowed with a linear connection $\nabla : TQ \times TQ \to TQ$. For vector fields $X, Y \in \mathfrak{X}(Q)$ the Hessian of $f \in C^\infty(Q)$ is $\mathrm{Hess}^\nabla(f)(X, Y) = XY(f) - \nabla_X Y(f)$. This is bilinear in X and Y but not symmetric, unless ∇ is torsion-free.

Definition 1.2 (Drift, Martingale, Brownian motion)**.**

- Let Γ be a diffusion in Q with generator A. Then the *drift* of Γ with respect to ∇ is defined to be the first order part of A, which is determined by ∇. If A is of the form (1.3) then this is $X_0 + \tfrac{1}{2}\sum \nabla_{X_i} X_i$.
- According to [8, Chapter IV] the A-diffusion Γ is a *martingale* in (Q, ∇) if A is purely second order with respect to ∇, i.e., the ∇-drift vanishes. In [8] this is stated for torsion-free connections but it is noted that one can use the same definition for connections with torsion.
- If (Q, μ) is a Riemannian manifold then an A-diffusion is called *Brownian motion* if $A = \tfrac{1}{2}\Delta$ where Δ is the metric Laplacian.

As it stands, Brownian motion is not unique. There may be several diffusions in Q such that the associated system of diffusion measures is generated by $A = \frac{1}{2}\Delta$. However, since we have defined diffusions in terms of systems of diffusion measures we regard two diffusions which give rise to the same system of diffusion measures as equivalent. Now the system of diffusion measures generated by $A = \frac{1}{2}\Delta$ is unique and it is in this sense that we think of Brownian motion as being unique.

To construct Brownian motion in (Q, μ) consider the orthonormal frame bundle $\rho : \mathcal{F} \to Q$ over (Q, μ). The Levi-Civita connection on Q gives rise to a unique principal bundle connection ω on $\rho : \mathcal{F} \to Q$. An element $u \in \mathcal{F}$ can be regarded as an isometry $u : \mathbb{R}^d \to T_{\rho(u)}Q$ where $d = \dim Q$. Let (e_i) be the standard basis in $\mathbb{R}^d$. Define the canonical horizontal vector fields $L_i \in \mathfrak{X}(\mathcal{F}, \mathrm{Hor}^\omega)$, $i = 1, \ldots, d$, by

$$(1.4) \qquad L_i(u) = \mathrm{hl}_u^\omega(u(e_i))$$

where $\mathrm{hl}^\omega : \mathfrak{X}(Q) \to \mathfrak{X}(\mathcal{F})$ is the horizontal lift map of ω. If (W^i) is Brownian motion in $\mathbb{R}^d$ and Γ solves the Stratonovich equation

$$(1.5) \qquad \delta\Gamma = \sum L_i(\Gamma)\delta W^i$$

then $\rho \circ \Gamma$ is a diffusion in Q with generator $\frac{1}{2}\Delta^\mu$, that is, a Brownian motion. This is explained in [20, Chapter V.4] and follows also from Theorem 1.3 below; the essential observation in this context is that the Stratonovich operator $\mathcal{S}$ of (1.5) enjoys the equivariance relation

$$\mathcal{S}(ug, g^{-1}w, g^{-1}w') = \left(\mathrm{hl}^\omega(u)(u(ge_i))\langle e_i, g^{-1}e_i\rangle\right)g = \mathcal{S}(u, w, w')g$$

for the principal right action of the structure group. To connect with Theorem 1.3 the principal right action can be turned to a left action via inversion in the group.

1.B. **Equivariant reduction.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, Q, $X_0, X_1, \ldots, X_k \in \mathfrak{X}(Q)$ and $\delta\Gamma = \mathcal{S}(Y, \Gamma)\delta Y$ as before. Suppose there is a Lie group G which acts properly on Q from the left. We extend this action to $\dot{Q}$ by requiring ∞ to be a fixed point. Let $\pi : Q \twoheadrightarrow Q/G$ be the projection and $C^\infty(Q)^G$ denote the subspace of G-invariant smooth functions on Q. Note that Q/G need not be a manifold; in general Q/G is a topological space which is naturally stratified by smooth components.

In the following all actions are tangent lifted where appropriate without further notice.

The proof of the following theorem is sketched in the appendix.

Theorem 1.3 (Eqivariant reduction [19])**.** *Suppose there is a group representation $\rho : G \to \mathrm{O}(k)$ and let $\mathrm{O}(k)$ act on $\mathbb{R}^{k+1} = \mathbb{R} \times \mathbb{R}^k$ such that the first factor is acted upon trivially. If $\mathcal{S}$ satisfies the equivariance property*

$$(1.6) \qquad \mathcal{S}(gx, \rho(g)y, \rho(g)y') = g.\mathcal{S}(x, y, y')$$

for all $(x, y, y') \in Q \times T\mathbb{R}^{k+1}$ then the diffusion Γ induces a diffusion $\pi \circ \Gamma$ in Q/G. Moreover, A preserves $C^\infty(Q)^G$, and the induced generator A_0 of $\pi \circ \Gamma$ is characterized by

$$(1.7) \qquad \pi^* A_0 f = A\pi^* f$$

for $f \in C^\infty(Q/G) := \{f \in C(Q/G) : \pi^* f \in C^\infty(Q)^G\}$.

Equivariant reduction is a natural extension of the reduction theory of [26, Theorem 3.1]. While the results of [26] are stronger in the sense that they provide a Stratonovich equation on the base space Q/G, they are only applicable when the original Stratonovich operator is G-invariant. This means that

$$(1.8) \qquad \mathcal{S}(gx, y, y') = g.\mathcal{S}(x, y, y')$$

for all $g \in G$ and $x \in Q$ and [26] show how to obtain an induced Stratonovich operator on the base space Q/G. This covers the case where the Stratonovich operator is defined in terms of G-invariant vector fields. To connect with the above theorem, note that a G-invariant operator $\mathcal{S}$ can be considered G-equivariant with respect to the trivial action on the source space $\mathbb{R}^{k+1}$, i.e., where $\rho(g)y = y$ for all $g \in G$ and $y \in \mathbb{R}^{k+1}$. By contrast the observation in equivariant reduction is that although the upstairs Stratonovich operator $\mathcal{S}$ is not projectable to Q/G, the diffusion still factors to a diffusion in the base space and the generator of the downstairs diffusion is induced from that of the original diffusion on Q.

2. The (almost) Hamiltonian construction of Brownian motion and symmetry reduction

2.A. **General version.** Let (Q, μ) be a Riemannian manifold, G a Lie group acting properly and through isometries on Q. We do not require the action to be free. Assume $\mathcal{H} : T^*Q \to \mathbb{R}$ is a G-invariant Hamiltonian function (e.g., the kinetic energy associated to μ). We want to describe a perturbation by a Brownian motion of the Hamiltonian system $(T^*Q, \Omega^Q, \mathcal{H})$ with dynamics $X_{\mathcal{H}}$; here $X_{\mathcal{H}}$ is the Hamiltonian vector field with respect to the canonical exact symplectic form Ω^Q on T^*Q. If $Q = \mathbb{R}^n$ was a vector space and μ the Euclidean metric then this should give rise to an Ito equation of the form

$$(2.9) \qquad d\Gamma = X_{\mathcal{H}}(\Gamma)dt + dW$$

with W Brownian motion in Q. This is Bismut's notion of a Hamiltonian diffusion described in [4]. Lazaro-Cami and Ortega [25] have generalized this set-up to non-Euclidean spaces and incorporated general semi-martingales as source of the noise.

To generalize (2.9) to the non-Euclidean setting one notices that it defines a diffusion with generator $X_{\mathcal{H}} + \frac{1}{2}\Delta^\mu$ where $\Delta^\mu = \sum \partial_i \partial_i$ is the Laplace-Beltrami operator on (Q, μ) viewed as acting on $C^\infty(T^*Q)$, thus identifying $\partial_i = \frac{\partial}{\partial q^i}$ with $(\partial_i, 0) = \sum \frac{\partial H^i}{\partial p_i} \frac{\partial}{\partial q^i}$ which is the Hamiltonian vector field of $H^i : T^*Q \to \mathbb{R}$, $(q, p) \mapsto \langle p, e_i \rangle$ with $\frac{\partial}{\partial q^i} = e_i$ the standard basis. Below we will give a general construction of such a diffusion. In doing so we follow essentially the ideas of [25].

Thus we consider the orthonormal frame bundle $\rho : P \to Q$ and denote its structure group by $K = O(n)$, $n = \dim(Q)$. The Levi-Civita connection ∇ on (Q, μ) induces a unique principal bundle connection form $\omega : TP \to \mathfrak{k}$. We use ω to decompose $TQ = \mathrm{Hor}^\omega \oplus \mathrm{Ver}^\rho$ into horizontal space $\mathrm{Hor}^\omega = \ker \omega$ and vertical space $\mathrm{Ver}^\rho = \ker T\rho$. The soldering form $\sigma \in \Omega^1(P, \mathbb{R}^n)$ is defined by $\sigma(\xi_u) = u^{-1}(T_u\rho.\xi)$ for $\xi_u \in T_uP$, and this induces a trivialization of the horizontal bundle. Therefore, we obtain a trivialization of TP by

$$(2.10) \qquad \Phi : TP \longrightarrow P \times \mathbb{R}^n \times \mathfrak{k}, \quad \xi_u \longmapsto \Big(u, \sigma(\xi_u), \omega(\xi_u)\Big)$$

We may use this isomorphism to obtain a metric μ^P on P by using the standard inner products on $\mathbb{R}^n$ and $\mathfrak{k}$ and requiring Φ to be an isometry. If $e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$ then we may use Φ to define vector fields $L_i \in \mathfrak{X}(P, \mathrm{Hor}^\omega)$ by

$$L_i(u) = \Phi^{-1}(u, e_i, 0) = \mathrm{hl}_u^\omega(u(e_i))$$

where $\mathrm{hl}^\omega : P \times_Q TQ \to \mathrm{Hor}^\omega$ is the horizontal lift mapping. The L_i are called the canonical horizontal vector fields. Notice that they form a global orthonormal frame for Hor^ω.

We remark that μ^P is K-invariant and the K-invariant subspaces Hor^ω and Ver^ρ are perpendicular with respect to μ^P. By construction ω coincides with the mechanical connection (defined in (3.16)) on (P, μ^P).

Furthermore, since G acts by isometries on Q it induces an action on P which also preserves μ^P: an element $u \in P$ is viewed as an isometry $u : \mathbb{R}^n \to T_{\rho(u)}Q$ and $g \in G$ acts from the left via $g.u = T_{\rho(u)}g \circ u : \mathbb{R}^n \to T_{g\rho(u)}Q$.

Now the principal action of K on P is a right action and to convert it to a left action we use inversion in the group. The actions of K and G commute whence the product $K \times G$ also acts on P. The cotangent lifted action by $K \times G$ is Hamiltonian. Singular Reduction in Stages thus implies that

$$T^*P /\!/_{0 \times \mathcal{O}} (K \times G) = (T^*P /\!/_0 K) /\!/_{\mathcal{O}} G = T^*Q /\!/_{\mathcal{O}} G$$

as stratified symplectic spaces; here $/\!/_{\mathcal{O}}$ denotes symplectic reduction at the coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ with respect to the obvious momentum map. See [28]. Let $\mathcal{H}^P := \rho^*\mathcal{H}$ be the $K \times G$-invariant upstairs Hamiltonian where $\rho : T^*P \to T^*Q$ is defined in terms of the metric isomorphisms $T^*P \cong_{\mu^P} TP$ and $T^*Q \cong_{\mu^Q} TQ$. Then (regular) Hamiltonian reduction of $(T^*P, \Omega^P, \mathcal{H})$ at $0 \in \mathfrak{k}^*$ with respect to K yields the original Hamiltonian system $(T^*Q, \Omega^Q, \mathcal{H})$. Consequently (singular) Hamiltonian reduction at $0 \times \mathcal{O} \subset \mathfrak{k}^* \times \mathfrak{g}^*$ of $(T^*P, \Omega^P, \mathcal{H}^P)$ is equivalent to (singular) Hamiltonian reduction of $(T^*Q, \Omega^Q, \mathcal{H})$ at $\mathcal{O} \subset \mathfrak{g}^*$.

Let us define the momentum functions

$$H^i : T^*P \longrightarrow \mathbb{R}, \quad \eta_u \longmapsto \langle \eta_u, L_i(u) \rangle$$

and the associated Hamiltonian vector fields X_{H^i}. Note that $T\tau_P.X_{H^i}(\eta_u) = L_i(u)$ where $\tau_P : T^*P \to P$ is the tangent projection. Therefore, if Γ^{T^*P} satisfies $\delta\Gamma^{T^*P} = \sum X_{H^i}(\Gamma^{T^*P})\delta W^i$ the projection $\tau_P \circ \Gamma^{T^*P}$ satisfies

(1.5), and $\tau_P \circ \Gamma^{T^*P}$ is Brownian motion in (Q, μ). This is the Hamiltonian construction of Brownian motion of [25].

Consider the Stratonovich operator

$$\mathcal{S}^{T^*P} : T^*P \times T(\mathbb{R} \times \mathbb{R}^n) \longrightarrow T(T^*P), \quad (\eta_u, t, t', w, w') \longmapsto X_{\rho^*\mathcal{H}}(\eta_u)t' + \sum X_{H^i}(\eta_u)\langle e_i, w'\rangle.$$

Let K act on $\mathbb{R} \times \mathbb{R}^n$ by acting on the second factor only and G act trivially on $R \times \mathbb{R}^n$. Thus $K \times G$ acts on $\mathbb{R} \times \mathbb{R}^n$.

Let $J_K : T^*P \to \mathfrak{k}^*$ denote the standard momentum map of the cotangent lifted K-action on T^*P. Then

$$J_K^{-1}(0) = \{\lambda \in T^*P : \langle \lambda, X\rangle = 0 \text{ for all } X \in \text{Ver}^\rho\} = \text{Hor}^*$$

where Hor^* is called the dual horizontal bundle. Contrary to Hor^ω the dual horizontal Hor^* is defined connection independently. Further, let $J_G : T^*P \to \mathfrak{g}^*$ denote the momentum map of the lifted G-action on T^*P, and correspondingly $J_{K \times G} = (J_K, J_G) : T^*P \to \mathfrak{k}^* \times \mathfrak{g}^*$ for the product action of $K \times G$.

Lemma 2.1. *The following hold.*

(1) *The operator $\mathcal{S}^{T^*P}$ is $K \times G$-equivariant.*
(2) *If $\mathcal{O} \subset \mathfrak{g}^*$ is a codjoint orbit, then $\mathcal{S}^{T^*P}$ restricts to a Stratonovich operator on $J_G^{-1}(\mathcal{O})$ in the sense that*

$$\mathcal{S}^{T^*P} : J_G^{-1}(\mathcal{O}) \times T(\mathbb{R} \times \mathbb{R}^n) \longrightarrow T\Big(J_G^{-1}(\mathcal{O})\Big).$$

If the G-action is non-free then the last statement holds for each smooth stratum in $J_G^{-1}(\mathcal{O})$.

Proof. (1) Clearly $X_{\rho^*\mathcal{H}}$ is $K \times G$-invariant. Furthermore, we have, for $g \in G$,

$$H^i(g.\eta_u) = \langle g.\eta_u, L_i(g.u)\rangle = \langle g.\eta_u, g.L_i(u)\rangle = \langle \eta_u, L_i(u)\rangle = H^i(\eta_u)$$

whence H^i and thus also X_{H^i} are G-invariant. This establishes G-equivariance of $\mathcal{S}^{T^*P}$.

To see K-equivariance we start by defining

$$H_{w'} : T^*P \longrightarrow \mathbb{R}, \quad \eta_u \longmapsto \sum H^i(\eta_u)\langle e_i, w'\rangle = \sum \langle \eta_u, \text{hl}_u^\omega\Big(u(w')\Big)\rangle$$

for $w' \in \mathbb{R}^n$. We note that

$$H_{k.w'}(k.\eta_u) = \sum \langle k.\eta_u, \text{hl}_{k.u}^\omega\Big(u(\langle e_i, k.w'\rangle k^{-1}e_i)\Big)\rangle = \sum \langle k.\eta_u, k.\text{hl}_u^\omega\Big(u(w')\Big)\rangle = H_{w'}(\eta_u)$$

Therefore, if we fix a vector $Y_{\eta_u} \in T_{\eta_u}(T^*P)$ and choose a (local) K-invariant vector field $Y \in \mathfrak{X}(T^*P)$ such that $Y(\eta_u) = Y_{\eta_u}$, we find

$$d\Big(\sum H^i\langle e_i, k.w'\rangle\Big)(k.\eta_u).T_{\eta_u}k.Y_{\eta_u} = \tfrac{\partial}{\partial t}|_0\Big((\text{Fl}_t^Y)^*H_{k.w'}\Big)(k.\eta_u) = \tfrac{\partial}{\partial t}|_0 H_{k.w'}(\text{Fl}_t^Y(k.\eta_u))$$

$$= \tfrac{\partial}{\partial t}|_0 H_{k.w'}(k.\text{Fl}_t^Y(\eta_u)) = \tfrac{\partial}{\partial t}|_0 H_{w'}(\text{Fl}_t^Y(\eta_u))$$

$$= d\Big(\sum H^i\langle e_i, w'\rangle\Big)(\eta_u).Y_{\eta_u}$$

Putting this together yields

$$\sum X_{H^i}(k.\eta_u)\langle e_i, k.w'\rangle = (\Omega^P)_{k.\eta_u}^{-1}\Big(d\Big(\sum H^i\langle e_i, k.w'\rangle\Big)(k.\eta_u)\Big)$$

$$= T_{\eta_u}k.(\Omega^P)_{\eta_u}^{-1}\Big(d\Big(\sum H^i\langle e_i, k.w'\rangle\Big)(k.\eta_u) \circ T_{\eta_u}k\Big)$$

$$= T_{\eta_u}k.(\Omega^P)_{\eta_u}^{-1}\Big(d\Big(\sum H^i\langle e_i, w'\rangle\Big)(\eta_u)\Big)$$

$$= T_{\eta_u}k.\sum X_{H^i}(\eta_u)\langle e_i, w'\rangle$$

whence K-equivariance of $\mathcal{S}^{T^*P}$ also follows.

(2). Since the H^i are G-invariant Noether's Theorem implies that the Hamiltonian vector fields used in the definition of $\mathcal{S}^{T^*P}$ are tangent to $J_G^{-1}(\mathcal{O})$. This is also true in the stratified context and moreover strata are preserved by Hamiltonian vector fields of invariant functions. See [37, 34]. $\square$

Now to reduce the Hamiltonian diffusion to $T^*Q /\!/_{\mathcal{O}} G$ we should check that $\mathcal{S}^{T^*P}$ restricts also to a Stratonovich operator on $J_K^{-1}(0) = \text{Hor}^* \subset T^*P$. For then, by equivariant reduction, we would obtain a diffusion in $(J_K^{-1}(0) \cap J_G^{-1}(\mathcal{O}))/(K \times G) = J_{K \times G}^{-1}(0 \times \mathcal{O})/(K \times G) = T^*Q /\!/_{\mathcal{O}} G$. Thus for $\eta_u \in \text{Hor}^*$ we should show that $\mathcal{S}^{T^*P}(\eta_u, w, w') \in T_{\eta_u}(J_K^{-1}(0)) = \ker dJ_K(\eta_u)$; that is, for $Y \in \mathfrak{k}$ and with ζ_Y^{K,T^*P} denoting the fundamental vector field associated to the cotangent lifted K-action on T^*P,

$$dJ_K. \sum X_{H^i}(\eta_u)\langle e_i, w'\rangle = -\sum dH^i(\eta_u).\zeta_Y^{K,T^*P}(\eta_u)\langle e_i, w'\rangle = -\tfrac{\partial}{\partial t}|_0 H_{w'}(\exp(tY).\eta_u)\langle e_i, w'\rangle$$
$$= -\tfrac{\partial}{\partial t}|_0 H_{w'}(\eta_u)\langle e_i, \exp(-tY).w'\rangle = H_{w'}(\eta_u)\langle e_i, Y.w'\rangle = \langle \eta_u, \text{hl}_u^\omega(w')\rangle.$$

However, this expression is certainly not 0 for general η_u, Y and w'.

Therefore, while $\rho \circ \tau \circ \Gamma^{T^*P}$ yields Brownian motion in (Q, μ) (see above), we *do not obtain an induced diffusion* in $T^*Q = \text{Hor}^*/K$. This problem can be resolved reflecting on the nature of the construction of Brownian motion in the Riemannian manifold Q: A Brownian path in Q is constructing by rolling Q along a Brownian path in $\mathbb{R}^n$ *without slipping or twisting*, i.e., rubber rolling on a rough surface. Thus we are dealing with a *non-holonomic* control problem. The configuration space of this problem is

$$\widetilde{Q} := \{(x, w, u) : x \in Q, w \in \mathbb{R}^n, u : T_xQ \to \mathbb{R}^n \text{ is an isometry}\} \cong P \times \mathbb{R}^n$$

and the constraint distribution which specifies the set of allowed motions is

$$\widetilde{\mathcal{D}} := \{(X_u, w, w') \in T(P \times \mathbb{R}^n) : w' = \sigma(X_u) \ \& \ X_u \in \text{Hor}_u^\omega\} \cong \text{Hor}^\omega \times \mathbb{R}^n$$

where $\sigma \in \Omega^1(P)$ is the soldering form as above. The abelian group $\mathbb{R}^n$ acts on the pair $(\widetilde{Q}, \widetilde{\mathcal{D}})$ by so-called outer symmetries which reflects the fact that the constraints are the same at all points of the "surface" $\mathbb{R}^n$ along which the rolling takes place. Thus we can reduce by the abelian $\mathbb{R}^n$ action and obtain a new configuration space - distribution pair $(P, \mathcal{D} := \text{Hor}^\omega)$. Suppose $\xi_i(t) : \Omega \to \mathbb{R}$ are a set of time-dependent controls. (This corresponds to the Brownian noise.) Then the non-holonomic control problem

$$(2.11) \qquad\qquad u'(t) = \sum L_i(u(t))\xi_i(t)$$

has an almost Hamiltonian formulation which is given by the ODE on $\mathcal{M} := \mu(\mathcal{D}) = \mu(\text{Hor}^\omega) = \text{Hor}^* = J_K^{-1}(0)$

$$\gamma'(t) = \sum \mathcal{P}(\gamma(t))X_{H^i}(\gamma(t))\xi_i(t)$$

(see [7, 5]) where $\mathcal{P} : T(T^*P)|\mathcal{M} \to \mathcal{C} \subset T\mathcal{M}$ is the Hamiltonian encoding of the *constraint force projection operator* (2.12).[2] With $\tau : T^*P \to P$ being the footpoint projection the space $\mathcal{C}$ is defined as

$$\mathcal{C} := \{Z \in T\mathcal{M} : T\tau.Z \in \mathcal{D}_{\tau(Z)} = \mathcal{D} \cap T_{\tau(Z)}P\}.$$

Now it can be shown that $T(T^*P)|\mathcal{M} = \mathcal{C} \oplus \mathcal{C}^\Omega$ where $\mathcal{C}^\Omega$ is the Ω-orthogonal of $\mathcal{C}$ in $T(T^*P)|\mathcal{M}$ with respect to the canonical symplectic form Ω on T^*P. Then

$$(2.12) \qquad\qquad \mathcal{P} : T(T^*P)|\mathcal{M} \longrightarrow \mathcal{C} \subset T\mathcal{M}$$

is defined as the projection along $\mathcal{C}^\Omega$. See [3]. Let $\Pi : TP \to \text{Hor}^\omega$ be the projection along the vertical space. It follows that $T\tau \circ \mathcal{P} = \Pi \circ \tau$. (See [19, Section 2]). Since $\Pi.L_i = L_i$ and X_{H^i} is τ-related to L_i one sees that

$$(2.13) \qquad\qquad u' = T\tau.\gamma' = T\tau.\mathcal{P}. \sum X_{H^i}(\gamma)\xi_i = \sum L_i(u)\xi_i$$

whence at this projected level one cannot distinguish the Hamiltonian from the non-holonomic version of the control problem (2.11). However, accepting the non-holonomic nature of the construction of Brownian motion one can give an almost Hamiltonian description in terms of the diffusion $\Gamma^{\mathcal{C}}$ in $\mathcal{M}$ defined by the Stratonovich equation

$$\delta\Gamma_t^{\mathcal{C}} = \mathcal{S}^{\mathcal{C}}(\Gamma_t, W_t, \delta W_t), \qquad \Gamma_0 = \eta_u \in \mathcal{M} \text{ a.s.}$$

where the $\mathcal{C}$-valued Stratonovich operator is given by

$$(2.14) \qquad \mathcal{S}^{\mathcal{C}} : \mathcal{M} \times T\mathbb{R}^n \longrightarrow \mathcal{C} \subset T\mathcal{M}, \quad (\eta_u, w, w') \longmapsto \sum \mathcal{P}(\eta_u)X_{H^i}(\eta_u)\langle e_i, w'\rangle.$$

---

[2]The notation $\mathcal{M}$ and $\mathcal{C}$ is introduced in order to be consistent with the non-holonomic literature such as [3].

Of course, by (2.13), we retain the basic feature that $\rho \circ \tau \circ \Gamma^{\mathcal{C}}$ is Brownian motion in Q. But now we also obtain an induced diffusion in T^*Q. Let $\pi^{T^*Q} : T^*Q \twoheadrightarrow (T^*Q)/G$ and $\pi^{\mathcal{M}} : \mathcal{M} \twoheadrightarrow \mathcal{M}/K = T^*Q$ be the orbit projections.

Proposition 2.2. *The diffusion $\Gamma^{\mathcal{C}}$ induces a diffusion $\Gamma^{T^*Q} = \pi^{\mathcal{M}} \circ \Gamma^{\mathcal{C}}$ in $\mathcal{M}/K = J_K^{-1}(0)/K = T^*Q$ as well as a diffusion $\Gamma^{\mathrm{red}} = \pi^{T^*Q} \circ \Gamma^{T^*Q}$ in $\mathcal{M}/(K \times G) = (T^*Q)/G$. Moreover, $\tau \circ \Gamma^{T^*Q}$ is Brownian motion in (Q, μ).*

Proof. Since $\mathcal{S}^{\mathcal{C}} = \mathcal{P}.\mathcal{S}^{T^*P}$ and the projection $\mathcal{P}$ is K-equivariant we preserve the K-equivariance of $\mathcal{S}^{T^*P}$. It is also easy to see that $\mathcal{P}$ is G-equivariant whence G-invariance of $\mathcal{S}^{T^*P}$ is preserved as well. Hence Theorem 1.3 applies to the K-action as well as to the $K \times G$-action on $\mathcal{M}$. $\qquad\square$

However, it is not clear that the induced diffusion Γ^{red} will preserve the symplectic leaves $T^*Q /\!/_{\mathcal{O}} G \subset (T^*Q)/G$.[3] The problem is that, in non-holonomic mechanics, symmetries need not lead to conservation laws, and correspondingly it cannot be asserted that $\mathcal{S}^{\mathcal{C}}(\eta_u, w, w') \in T_{\eta_u}(J_{K \times G}^{-1}(0 \times \mathcal{O}))$ for $\eta_u \in J_{K \times G}^{-1}(0 \times \mathcal{O})$. Instead of a conservation law one obtains a momentum equation, see [6].

2.B. **The flat case.** If the manifold (Q, μ) is flat so that it admits a global orthonormal frame $L_a \in \mathfrak{X}(Q)$, $a = 1, \ldots, n$ then one can give a Hamiltonian construction of Brownian motion in Q which is formulated on T^*Q ([25]): Define Hamiltonian momentum functions on T^*Q by

$$H^0(q, p) = -\tfrac{1}{2}\langle p, \sum \nabla_{L_a}^{\mu} L_a(q) \rangle \text{ and } H^a(q, p) = \langle p, L_a(q) \rangle.$$

Consider the Stratonovich equation $\delta \Gamma^{T^*Q} = \mathcal{S}^{T^*Q}(\Gamma^{T^*Q}, W, \delta W)$ where W is Brownian motion in $\mathbb{R}^n$ and the Stratonovich operator is defined as

$$(2.15) \quad \mathcal{S}^{T^*Q} : T^*Q \times T(\mathbb{R} \times \mathbb{R}^n) \longrightarrow T(T^*Q), \quad (q, p; t, t', w, w') \longmapsto X_{H^0}(q, p)t' + \sum X_{H^a}(q, p)\langle e_a, w' \rangle.$$

Then $\Gamma^Q := \tau \circ \Gamma^{T^*Q}$ is a diffusion in Q with generator $\tfrac{1}{2} \sum (L_a L_a - \nabla_{L_a}^{\mu} L_a) = \tfrac{1}{2}\Delta^{\mu}$, i.e., Brownian motion. If a Lie group G acts properly and by isometries on (Q, μ) and the frame (L_a) is G-invariant then Γ^{T^*Q} drops to a diffusion in the (singular) Poisson quotient $(T^*Q)/G$ which preserves the symplectic leaves $T^*Q /\!/_{\mathcal{O}} G = J_G^{-1}(\mathcal{O})/G$.

3. Reduction of Brownian motion with respect to polar actions

We give two different Hamiltonian constructions of Brownian motion on a Riemannian G-manifold (Q, μ), and discuss their respective symmetry reductions. Throughout T^*Q will be endowed with the canonical exact symplectic form denoted by Ω^Q or Ω if no confusion is possible.

3.A. **Generalities on transformation groups.** Consider a proper Riemannian G-manifold (Q, μ). This means that G is a Lie group acting properly and by isometries on the Riemannian manifold Q. We will assume that the G-action is of *single orbit type* which means that there is a subgroup $H \subset G$ such that, for any $q \in Q$, the stabilizer subgroup G_q is conjugate to H within G. This has the consequence that $B := Q/G$ is a smooth manifold and the projection map $\pi : Q \twoheadrightarrow B$ is a fiber bundle, albeit not a principle one. Its typical fiber is of the form G/H. See [35].

The vertical bundle $\mathrm{Ver} \subset TQ$ on $\pi : Q \twoheadrightarrow B$ is defined as

$$\mathrm{Ver} := \ker T\pi.$$

Definition 3.1. The (generalized) *mechanical connection* on $\pi : Q \twoheadrightarrow B$ is the fiber bundle connection which is specified by requiring the horizontal bundle Hor to be orthogonal to the vertical one with respect to the G-invariant Riemannian metric μ.[4]

---

[3]It is slightly imprecise to speak of symplectic leaves in this context: The $T^*Q /\!/_{\mathcal{O}} G$ need not be connected, they need not even be smooth manifolds. It is rather the connected components of the smooth strata of $T^*Q /\!/_{\mathcal{O}} G$ which should be called symplectic leaves.

[4]We have included the prefix 'generalized' because usually the mechanical connection is defined on a principal bundle. In order not to overload the nomenclature we will subsequently omit this prefix.

See (3.17) below for an alternative characterization of the mechanical connection.

From now on we equip $\pi : Q \twoheadrightarrow B$ with the mechanical connection. Let $\mathcal{A} : TQ \to \mathfrak{g}$ be the associated connection form. This is not a principal bundle connection form. For $q \in Q$ with isotropy group G_q we have

$$A_q \zeta_X(q) = X \text{ for all } X \in \mathfrak{g}_q^\perp.$$

Here $\mathfrak{g}_q \subset \mathfrak{g}$ is the infinitesimal stabilizer at q, $\zeta : \mathfrak{g} \to \mathfrak{X}(Q)$ is the fundamental vector field mapping, $\mathfrak{g}$ is equipped with an Ad-invariant inner product $\langle .,. \rangle$ and $\mathfrak{g}_q^\perp$ is the $\langle .,. \rangle$-orthogonal to $\mathfrak{g}_q$. At a point $q \in Q$ we have $\zeta_X(q) = \frac{\partial}{\partial t}|_0 \exp(tX).q \in T_qQ$. We may view A also as a G-equivariant bundle map

$$(3.16) \qquad\qquad A : TQ \longrightarrow \bigsqcup_{q \in Q} \mathfrak{g}_q^\perp$$

which restricts to an isomorphism on the vertical bundle. See [29] for a definition in the context of a free action and [15] for the generalization to single orbit type actions. Let $\xi_i = v_i + \zeta_{X_i} \in TQ = \mathrm{Hor} \oplus \mathrm{Ver}$ be decomposed into horizontal and vertical parts. Then

$$\mu(\xi_1, \xi_2) = \mu(v_1, v_2) + \mathbb{I}_q(X_1, X_2)$$

where the G-invariant operator $\mathbb{I} : \bigsqcup_{q \in Q} \mathfrak{g}_q^\perp \times \bigsqcup_{q \in Q} \mathfrak{g}_q^\perp \to \mathbb{R}$ is defined by

$$\mathbb{I}_q(X_1, X_2) := \mu_q(\zeta_{X_1}(q), \zeta_{X_2}(q)) = \langle A_q^* A_q X_1, X_2 \rangle,$$

and is called the INERTIA TENSOR. For each $q \in Q$ we obtain a G_q-equivariant isomorphism $\mathbb{I}_q : \mathfrak{g}_q^\perp \to \mathrm{Ann}\,\mathfrak{g}_q$ where $\mathrm{Ann}\,\mathfrak{g}_q$ is the annihilator of $\mathfrak{g}_q$ in $\mathfrak{g}^*$.

Definition 3.2 (The canonical momentum map on T^*Q). The map $J : T^*Q \to \mathfrak{g}^*$ defined by

$$\langle J(q,p), X \rangle = \langle p, \zeta_X(q) \rangle$$

is the *cotangent bundle momentum map*.

The mechanical connection A can now be characterized by the pointwise diagram

$$(3.17) \qquad\qquad \begin{array}{ccc} T_qQ & \xrightarrow{\;A_q\;} & \mathfrak{g}_q^\perp \\[2pt] {\scriptstyle\cong}\downarrow{\scriptstyle\mu_q} & & {\scriptstyle\mathbb{I}_q}\downarrow{\scriptstyle\cong} \\[2pt] T_q^*Q & \xrightarrow{\;J_q\;} & \mathrm{Ann}\,\mathfrak{g}_q \end{array}$$

where J_q is the restriction of J to T_q^*Q, whence its importance for mechanical systems.

Let us now assume additionally that G is compact and acts by polar transformations on Q. This means that there is a submanifold $M \subset Q$ which meets all group orbits in an orthogonal manner. The submanifold M is called a (global cross-) section of the action. See [35]. A canonical example is the conjugation action of a compact Lie group on itself. In this case a section is given by a maximal torus. As with this example, it is not required that the action be free. Note that there is a residual action by $W := \{g \in G : g.M \subset M\}/\{g \in G : g.x = x \; \forall x \in M\}$ on M and that

$$Q/G = M/W =: B.$$

In the aforementioned example W coincides with the Weyl group of the compact Lie group and B is the interior of a Weyl chamber.

We cocnctinue to assume that the G-action is of single orbit type. In the canonical example where the group acts upon itself by conjugation this amounts to passing to the open dense subset of regular points.

There is thus a local diffeomorphism $Q \cong M \times G/H$. This diffeomorphism is generally not global: Consider $SO(3)$ acting on $Q = \mathbb{R}^3 \setminus 0$ then $M = \mathbb{R} \setminus 0$ and $G/H = S^2$. Factoring out the residual W-action one does obtain a global diffeomorphism

$$Q \cong M \times_W G/H.$$

By reason of dimension it follows that W is discrete and hence finite since G is compact. Since $Q/G = M/W = B$ it is also true that $M \cong \bigsqcup_{w \in W} B$. Thus there is also a (non-canonical) diffeomorphism

$$(3.18) \qquad\qquad Q \cong B \times G/H.$$

Moreover, we will assume that the action is actually hyper-polar: M is supposed to be locally isometrically diffeomorphic to a Euclidean space $\mathbb{R}^l$ such that $B \cong \mathbb{R}^l$.

3.B. **Deterministic Hamiltonian reduction.** Let $\mathcal{H} : T^*Q \to \mathbb{R}$ be the G-invariant kinetic energy Hamiltonian associated to μ. By [15] Hamiltonian reduction of $(T^*Q, \Omega^Q, \mathcal{H})$ at the orbit level $\mathcal{O}$ yields the reduced stratified Hamiltonian system $(T^*Q /\!/_{\mathcal{O}} G, \Omega^{\mathcal{O}}, \mathcal{H}^{\mathcal{O}})$; using the mechanical connection A the phase space of this system can be realized as

$$(3.19) \qquad T^*Q /\!/_{\mathcal{O}} G \cong T^*B \times_B (\textstyle\bigsqcup_{q \in Q} \mathcal{O} \cap \operatorname{Ann} \mathfrak{g}_q)/G, \quad [(q,p)] \longmapsto (\pi(q), \eta_q^*(p); [(q, A_q^*(\lambda))])$$

where $\eta_q : T_x B \to T_q Q$ is the horizontal lift mapping associated to A. The reduced Hamiltonian then becomes

$$(3.20) \qquad\qquad \mathcal{H}^{\mathcal{O}} : (x = \pi(q), u; [(q, \lambda)]) \longmapsto \tfrac{1}{2}\langle u, u \rangle + \tfrac{1}{2}\langle \lambda, \mathbb{I}_q^{-1}(\lambda) \rangle.$$

The reduced space $T^*B \times_B (\bigsqcup_{q \in Q} \mathcal{O} \cap \operatorname{Ann} \mathfrak{g}_q)/G$ is a stratified symplectic fiber bundle over T^*B with standard fiber $(\mathcal{O} \cap \operatorname{Ann} \mathfrak{h})/H = \mathcal{O} /\!/_0 H$ which is a stratified symplectic space. This is the (singular) bundle picture in mechanics. The potential term $\tfrac{1}{2}\langle \lambda, \mathbb{I}_q^{-1}(\lambda) \rangle$ can also be viewed as a W-invariant function on $M \times \mathcal{O} /\!/_0 H$.

3.C. **Brownian motion in a constant frame and reduction.** The situation we have in mind in this subsection is that of a mechanical system defined on a Lie group G such that the (kinetic energy) Hamiltonian is invariant under the conjugation action and the tangent space is trivialized as $TG = G \times \mathfrak{g}$ via (left or right) multiplication. Prototypical examples for this set-up are the Calogero-Moser systems discussed in Section 4. This set-up is quite general in the sense that the hierarchy of Calogero-Moser systems is very rich; any (real or complex) semi-simple or also reductive Lie group can be taken as a configuration space, and different choices will lead to different (versions of Calogero-Moser) dynamical systems. The classical Calogero-Moser system corresponds to the choice $G = \operatorname{SU}(n)$ of [21]. Generalizations where G is a loop group could also be feasible and will give rise to a Calogero-Moser system with an elliptic interaction potential, see e.g. [24].

Suppose Q carries a global orthonormal frame L_A, $A = 1, \ldots, n$ such that $\nabla_{L_A}^\mu L_A = 0$ and $TQ \cong Q \times \mathfrak{q}$ via this frame with $\mathfrak{q} = \mathbb{R}^n$. A Hamiltonian construction Γ^{T^*Q} of Brownian motion $\Gamma^Q = \tau \circ \Gamma^{T^*Q}$ can then be given in terms of the Stratonovich equation associated to operator $\mathcal{S}^{T^*Q}$ defined in (2.15). Since $H_{w'}(q,p) = \sum H^A(q,p)\langle e_A, w' \rangle = \langle p, w' \rangle = H_{gw'}(gq, gp)$ for $g \in G$ it follows as in the proof of Lemma 2.1 that $\mathcal{S}^{T^*Q}$ is G-equivariant. Therefore, by Theorem 1.3 the diffusion Γ^{T^*Q} factors through the projection $\pi^{T^*Q} : T^*Q \twoheadrightarrow (T^*Q)/G$ to $\pi^{T^*Q} \circ \Gamma^{T^*Q}$ in the quotient. Consider the diagram

$$(3.21) \qquad\qquad \begin{array}{ccc} T^*Q & \xrightarrow{\;\pi^{T^*Q}\;} & (T^*Q)/G \\ {\scriptstyle \tau}\Big\downarrow & & \Big\downarrow{\scriptstyle \tau/G} \\ Q & \xrightarrow{\;\;\pi\;\;} & Q/G \end{array}$$

We can alternatively write Brownian motion on (Q, μ) as the solution to the Stratonovich diffusion equation $\delta\Gamma^Q = \sum L_A \delta W^A$. Again this operator is G-equivariant and, repeating the same reasoning, we obtain an induced diffusion

$$\Gamma^B = \pi \circ \Gamma^Q = (\tau/G) \circ \pi^{T^*Q} \circ \Gamma^{T^*Q}$$

in $B = Q/G$. In order to obtain the generator $\tfrac{1}{2}\Delta^0$ of Γ^B we note that the generator of Γ^Q is $\tfrac{1}{2}\Delta^Q$ and introduce the following function: Let e_α be a an orthonormal basis on $\mathfrak{g}$ and define $\delta : Q \to \mathbb{R}_{>0}$ by

$$(3.22) \qquad\qquad \delta(q) = \left| \det(\mathbb{I}_q(e_\alpha, e_\beta))_{\alpha\beta} \right|^{\frac{1}{2}}.$$

This function is G-invariant and we will call the induced function on B also δ. The projection $\pi : Q \twoheadrightarrow B$ is a Riemannian submersion with respect to the induced metric μ_0 on B. Let Δ^Q, Δ^B denote the Laplace-Beltrami operator on (Q, μ), (B, μ_0) respectively. Note that Δ^Q acts on the set of G-invariant functions $C^\infty(Q)^G$.

Proposition 3.3. *Let $\tfrac{1}{2}\Delta^0$ be the induced operator on $C^\infty(B)$ characterized by $\pi^* \circ \Delta^0 = \Delta^Q \circ \pi^*$. Then*

$$(3.23) \qquad\qquad \Delta^0 = \Delta^B + \nabla \log \delta$$

where ∇ is the μ_0-gradient.

Proof. This follows either from the coordinate expression $\Delta^Q = \sum g^{-\frac{1}{2}} \partial_i \sqrt{g} g^{ij} \partial_j$ in suitably choosen local coordinates, or by appealing to [14].   □

We view Γ^B as the reduction of the probabilistic version of a free particle motion on (Q, μ).

It is known that, if Γ^B is *critical in the sense of Guerra and Morato* for the stochastic action functional associated to the Lagrangian $L : TB \to \mathbb{R}$,

$$L(q, v) = \tfrac{1}{2}\mu_0(v, v) - V(q),$$

then, with $\hbar = 1$, the stochastic Hamilton-Jacobi equation of Nelson [30, p. 72, Equ. (14.17)]

$$(3.24) \qquad\qquad V - \tfrac{1}{2}\mu_0(b, b) - \tfrac{1}{2}\mathrm{div}\, b = 0$$

holds. Note that by (3.23) and Definition 1.2, $b = \tfrac{1}{2}\nabla(\log \delta)$ is the drift of the diffusion Γ^B (with respect to the Levi-Civita connection associated to μ_0.). See also Guerra and Morato [12], or Zambrini [39, Equ. (3.14)] for a version in terms of Yasue's least Action Principle. The drift b is a pure osmotic velocity in the nomenclature of [30].

Remark 3.4. We have arrived at (3.24) by asking the the following question: Is there a Lagrangian $L = \tfrac{1}{2}||\cdot||_{\mu_0} - V$ on TB such that the induced diffusion Γ^B is critical for L in the sense of Guerra-Morato? By [30, Thm. 14.1] this is equivalent to solving $v = \nabla S$ where v is the current velocity and

$$S(x, t) = -E\Big[\int_t^{t_1} L_+(\Gamma_s^B, s)\, ds \big| \Gamma_t^B = x \Big]$$

and $L_+(x, s) = \tfrac{1}{2}\mu_0(b, b)_x + \tfrac{1}{2}\mathrm{div}(b)_x - V(x, s)$. Given the diffusion (and its drift) this amounts to finding a suitable potential function V, and the simplest possible choice is expressed by (3.24). That is, $L_+ = 0$ and $v = 0$. Since the drift is the sum of osmotic and current velocity it remains to verify that $b = \tfrac{1}{2}\nabla(\log \delta)$ satisfies the osmotic equation

$$\tfrac{1}{2}\Delta\rho = \mathrm{div}(\rho b)$$

and this certainly holds with $\rho = \delta$ and $b = \tfrac{1}{2}\nabla(\log \delta)$. We emphasize that this corresponds to the simplest possible choice for a potential. There could be more interesting choices, arising (probably) from a spin-debendent potential.

Thus we use (3.24) as the defining equation for V and find

$$(3.25) \qquad\qquad V = \tfrac{1}{4}(\Delta^B \log \delta + \tfrac{1}{2}\mu_0(\nabla \log \delta, \nabla \log \delta) = \tfrac{1}{2}\delta^{-\frac{1}{2}}\Delta^B \delta^{\frac{1}{2}}$$

With

$$\psi = \delta^{\frac{1}{2}}$$

and $b = \tfrac{1}{2}\nabla(\log \delta)$ equation (3.24) is equivalent to the stationary Schrödinger equation $H\psi = 0$ with the Hamiltonian operator

$$(3.26) \qquad\qquad H = -\tfrac{1}{2}\Delta^B + V;$$

indeed

$$\begin{aligned}
\delta^{-\frac{1}{2}} H \delta^{\frac{1}{2}} &= \delta^{-\frac{1}{2}}\Big(-\tfrac{1}{2}\Delta\delta^{\frac{1}{2}} + V\delta^{\frac{1}{2}} \Big) = \delta^{-\frac{1}{2}}\Big(-\tfrac{1}{2}\mathrm{div}(\tfrac{1}{2}\delta^{-\frac{1}{2}}\nabla\delta) + V\delta^{\frac{1}{2}} \Big) \\
&= \tfrac{1}{8}\delta^{-2}\mu_0(\nabla\delta, \nabla\delta) - \delta^{-1}\Delta\delta + V = \tfrac{1}{2}\mu_0(b, b) - \tfrac{1}{4}\delta^{-2}\mu_0(\nabla\delta, \nabla\delta) - \tfrac{1}{4}\mathrm{div}(\nabla \log \delta) + V \\
&= -\tfrac{1}{2}\mu_0(b, b) - \tfrac{1}{2}\mathrm{div}(b) + V.
\end{aligned}$$

To compare these formulas with the case of quantum reduction of a free particle under polar actions we quote a special case of a theorem of [11, Thm. 4.5].

Theorem 3.5 (Feher and Pusztai [11], spinfree version)**.** *The reduction of the (spinless) quantum system defined by the closure of $-\tfrac{1}{2}\Delta^Q$ on $C^\infty_{\mathrm{cp}}(Q) \subset L^2(Q, d\mu)$ leads to the reduced Hamiltonian operator*

$$(3.27) \qquad\qquad H_{\mathrm{QM}} = H$$

with H as in (3.26).[5] *This operator is essentially self-adjoint on a suitable domain (specified in [11]).*

We note that this theorem, as it is stated, is actually only the zeroth order version of [11, Thm. 4.5] since in its full version it also incorporates a spin dependent potential energy term. In the next section we give a different approach where such a term can also be included in the stochastic context.

3.D. **Brownian motion in a G-adapted frame and reduction.** Assume the quotient $B = M/W = Q/G \cong \mathbb{R}^l$ is Euclidean and denote its x-independent orthonormal basis by

$$(3.28) \qquad v_1, \ldots, v_l.$$

Let

$$(3.29) \qquad Y_1(x), \ldots, Y_k(x)$$

denote an $\mathbb{I}_x$-orthonormal frame on $\mathfrak{h}^\perp \subset \mathfrak{g}$ which depends smoothly on $x \in M$. (Since $M = \bigsqcup_{w \in W} B$ as argued in Section 3.A we may non-canonically embed B in M as an open subset and thus we use the same variable x for elements in B as well as in M.) Concerning indices we make the convention that

$$1 \leq A, B, C \leq n = \dim Q, \quad 1 \leq i, j, k \leq l = \dim M, \quad 1 \leq \alpha, \beta, \gamma \leq k = \dim(G/H).$$

Then using (3.18) $(L_A)_A$ is an orthonormal frame on $Q \cong B \times G/H$ where we define

$$L_A(q) = L_A(g.x) = \begin{cases} g.v_i & \text{if } A = i; \\ g.\zeta_{Y_\alpha(x)}(x) & \text{if } A = \alpha + l. \end{cases}$$

Now Brownian motion in (Q, μ) can be constructed via $\Gamma^Q = \tau \circ \Gamma^{T^*Q}$ as in (2.15). Thus we need to calculate H^0: take local coordinates (q^A) around a point $q = g.x$ which are adapted to the decomposition $T_q Q \cong T_x M \oplus T_{[g]} G/H$ and express $L_A = \sum L_A^B \frac{\partial}{\partial q^b}$ whence

$$\sum \nabla_{L_A}^\mu L_A = \sum L_A^B \nabla_{\frac{\partial}{\partial q^B}}^\mu (L_A^C \frac{\partial}{\partial q^C}) = 0 + \sum L_A^B L_A^C \Gamma_{BC}^D \frac{\partial}{\partial q^D} = \sum \mu^{BC} \Gamma_{BC}^D \frac{\partial}{\partial q^D}$$

$$= -\frac{1}{\sqrt{|\mu|}} \frac{\partial(\sqrt{|\mu|} \mu^{DE})}{\partial q^D} \frac{\partial}{\partial q^D} = -\nabla \log \delta$$

where $\delta = \sqrt{|\mu|}$ was defined in (3.22). Therefore, $H^0(q, p) = \frac{1}{2}\langle p, \nabla \log \delta \rangle$.

The Hamiltonian diffusion Γ^{T^*Q} in T^*Q is defined by the Stratonovich equation

$$(3.30) \qquad \delta\Gamma^{T^*Q} = X_{H^0}(\Gamma^{\mathcal{H}})\delta t + \sum X_{H^A}(\Gamma^{\mathcal{H}})\delta W^A$$

with notation as in (2.15). The associated Stratonovich operator $T^*Q \times T(\mathbb{R} \times \mathbb{R}^n) \to T(T^*Q)$, $(q, p, t, t', w, w') \mapsto X_{H^0}(q, p)t' + \sum X_{H^A}(q, p)\langle e^A, w' \rangle$ is clearly G-*invariant* and reduces to an operator $\mathcal{S}^{\mathcal{O}}$ on the reduced phase space $T^*Q /\!/_{\mathcal{O}} G$. In particular, if $\pi^{T^*Q} : T^*Q \twoheadrightarrow (T^*Q)/G$ is the orbit space projection we have:

Proposition 3.6. Γ^{T^*Q} *drops to a diffusion* $\pi^{T^*Q} \circ \Gamma^{T^*Q}$ *in* $(T^*Q)/G$ *which, for every coadjoint orbit* $\mathcal{O} \subset \mathfrak{g}^*$, *preserves the smooth connected symplectic strata in* $T^*Q /\!/_{\mathcal{O}} G \subset (T^*Q)/G$.

Proof. On each smooth connected symplectic stratum $L \subset T^*Q /\!/_{\mathcal{O}} G$ we can express $\pi^{T^*Q} \circ \Gamma^{T^*Q}$ as the solution to the Hamiltonian Stratonovich equation $\delta(\pi^{T^*Q} \circ \Gamma^{T^*Q}) = \mathcal{S}^L(\pi^{T^*Q} \circ \Gamma^{T^*Q}, t, \delta t, W, \delta W)$. The reduced operator $\mathcal{S}^L$ on L is given by

$$\mathcal{S}^L : L \times T(\mathbb{R} \times \mathbb{R}^n) \longrightarrow TL, \quad (\eta, t, t', w, w') \longmapsto X_{h^0}(\eta)t' + \sum X_{h^A}(\eta)\langle e_A, w' \rangle$$

where h^0, h^A are the induced functions on L which pull-back to the restrictions of H^0, H^A to $(\pi^{T^*Q})^{-1}(L)$. See also [26]. $\qquad \square$

---

[5]The Planck constant $\hbar$ could be introduced by rescaling the diffusion operator on Q, i.e., the metric μ. This rescaling does not affect the potential V since factor cancels in this term.

Let us denote the restriction of $\pi^{T^*Q} \circ \Gamma^{T^*Q}$ to $T^*Q /\!/_{\mathcal{O}} G$ by $\Gamma^{\mathcal{O}}$. According to the above this diffusion further restricts to each smooth symplectic stratum $L \subset T^*Q /\!/_{\mathcal{O}} G$.

To obtain formulas for h^0 and h^A we use the A-dependent G-equivariant isomorphism

$$\Phi: \; \mathcal{W} := Q \times_B T^*B \oplus \bigsqcup_{q \in Q} \mathrm{Ann}\,\mathfrak{g}_q \longrightarrow T^*Q, \quad \big((q;x,u),(q,\lambda)\big) \longmapsto \big(q, (T_q^*\pi)(u) + A_q^*(\lambda)\big).$$

(This isomorphism depends on the choice of the connection and works only if the action is of single orbit type. However, it does not depend on whether or not the action is polar.) Thus we find, denoting the function on B induced from (3.22) again by δ,

$$\Phi^* H^0\big((q;x,u),(q,\lambda)\big) = \langle (T_q\pi)^*(u) + A_q^*(\lambda), -\tfrac{1}{2}\sum \nabla_{L_A}^\mu L_A \rangle = \tfrac{1}{2}\langle u, \nabla \log \delta(x)\rangle,$$
$$\Phi^* H^i\big((q;x,u),(q,\lambda)\big) = \langle u, v_i\rangle,$$
$$\Phi^* H^\alpha\big((q;x,u),(q,\lambda)\big) = \langle A_q^*(\lambda), g.\zeta_{Y_\alpha(x)}(x)\rangle = \langle g^{-1}.\lambda, Y_\alpha(x)\rangle$$

where we used the equivariance property of A, and x, q and g are related by $q = g.x$. The basis elements v_i and Y_α are introduced in (3.28) and (3.29).

Because we want to apply the stochastic Hamilton-Jacobi equation of Lazaro-Cami and Ortega [27] we want the reduced space $T^*Q /\!/_{\mathcal{O}} G$ to be a cotangent bundle. Therefore, we assume now that

$$\mathcal{O} /\!/_0 H = \{\text{point}\} = \{[\lambda]_H\}$$

whence $T^*Q /\!/_{\mathcal{O}} G = T^*B \times \{[\lambda]_H\} = T^*B$. This is a very restrictive condidtion and cases where this assumption holds are very important in the theory of Calogero-Moser systems. See [10] for a classification, further information and the so-called 'KKS-mechanism'. The reduced Hamiltonian functions thus become

$$h^0(x,u) = \tfrac{1}{2}\langle u, \nabla \log \delta(x)\rangle,$$
$$h^i(x,u) = \langle u, v_i\rangle,$$
$$h^\alpha(x,u) = \langle \lambda, Y_\alpha(x)\rangle.$$

Let $f \in C^\infty(T^*B)$. The stochastic Hamilton-Jacobi equation of [27] associated to the reduced Hamiltonian diffusion $\Gamma^{\mathcal{O}}$, given by

$$(3.31) \qquad \delta\Gamma^{\mathcal{O}} = X_{h^0}(\Gamma^{\mathcal{O}})\delta t + \sum X_{h^i}(\Gamma^{\mathcal{O}})\delta W^i + \sum X_{h^\alpha}(\Gamma^{\mathcal{O}})\delta W^\alpha,$$

and the Lagrangian submanifold $L_f = \mathrm{graph}\,df \subset T^*B$ is an equation of semi-martingales which reads

$$(3.32) \qquad \widetilde{S}_t^x(\omega) = f(x) - \int_0^t h^0(x, d\widetilde{S}_s^x(\omega))\,ds - \int_0^t h^i(x, d\widetilde{S}_s^x(\omega))\,\delta W_s^i - \int_0^t h^\alpha(x, d\widetilde{S}_s^x(\omega))\,\delta W_s^\alpha, \qquad \text{a.s.}$$

Here $\widetilde{S}^x$ is, for all $x \in B$, a continuous $\mathbb{R}$-valued semimartingale defined on some underlying probability space. The map $B \to \mathbb{R}$, $x \mapsto \widetilde{S}_t^x(\omega)$ is C^1 where it is well-defined and accordingly $d\widetilde{S}_t^x(\omega) = d(x \mapsto \widetilde{S}_t^x(\omega))$. These statements are proved in [27] where $\widetilde{S}$ is called the *projected stochastic action* and is constructed in terms of the defining data of the Hamiltonian diffusion and the Lagrange submanifold L_f. In the following lines we parallel the arguments of [27, Example 1] adapted to (3.32).[6] Thus we use the conversion rule [36, p. 81] that transforms Ito to Stratonovich equations and find, a.s.,

$$\widetilde{S}_t^x(\omega) = f(x) - \int_0^t \tfrac{1}{2}\langle \nabla \widetilde{S}_s^x(\omega), \nabla \log \delta\rangle\,ds - \sum \int_0^t \frac{\partial \widetilde{S}_s^x(\omega))}{\partial x^i}\,\delta W_s^i - \sum \langle \lambda, Y_\alpha(x)\rangle \int_0^t \delta W_s^\alpha$$
$$= f(x) + \int_0^t \big(\tfrac{1}{2}\Delta \widetilde{S}_s^x(\omega) - \tfrac{1}{2}\langle \nabla \widetilde{S}_s^x(\omega), \nabla \log \delta\rangle\big)\,ds - \sum \int_0^t \frac{\partial \widetilde{S}_s^x(\omega))}{\partial x^i}\,dW_s^i - \sum \langle \lambda, Y_\alpha(x)\rangle W_t^\alpha.$$

For the quadratic variation $[\widetilde{S}_t^x, \widetilde{S}_t^x]$ (see [36, Thm. II.29]) this implies

$$\tfrac{1}{2}[\widetilde{S}_t^x, \widetilde{S}_t^x] = \tfrac{1}{2}\int_0^t \big(\langle \nabla \widetilde{S}_s^x, \nabla \widetilde{S}_s^x\rangle + \sum \langle \lambda, Y_\alpha(x)\rangle^2\big)\,ds = \int_0^t \mathcal{H}^{\mathcal{O}}(x, \nabla \widetilde{S}_s^x)\,ds$$

---

[6]There are two small differences: We use the convention that $[W^i, W^j] = \delta^{ij}t$, and there is a constant factor in [27, Example 1] which we could not verify whence some other choices differ as well.

since $\sum \langle \lambda, Y_\alpha(x) \rangle^2 = \langle \lambda, \mathbb{I}_x^{-1}(\lambda) \rangle$ which equals twice the potential term (3.20) in the reduced deterministic Hamiltonian $\mathcal{H}^{\mathcal{O}}$. Applying the Ito formula [36, Thm. II.32] to the semi-martingale $\exp(-\widetilde{S}^x)$ thus yields

$$
\begin{aligned}
e^{-\widetilde{S}_t^x} - e^{-f(x)} &= -\int_0^t e^{-\widetilde{S}_s^x} \, d\widetilde{S}_s^x + \tfrac{1}{2}\int_0^t e^{-\widetilde{S}_s^x} \, d[\widetilde{S}_s^x, \widetilde{S}_s^x] \\
&= \int_0^t e^{-\widetilde{S}_s^x}\left(-\tfrac{1}{2}\Delta\widetilde{S}_s^x(\omega) + \tfrac{1}{2}\langle\nabla\widetilde{S}_s^x(\omega), \nabla\log\delta\rangle + \tfrac{1}{2}\langle\nabla\widetilde{S}_s^x(\omega), \nabla\widetilde{S}_s^x(\omega)\rangle + \tfrac{1}{2}\langle\lambda, \mathbb{I}_x^{-1}(\lambda)\rangle\right)ds \\
&\quad + \sum\int_0^t e^{\widetilde{S}_s^x}\frac{\partial\widetilde{S}_s^x(\omega))}{\partial x^i} \, dW_s^i + \sum\langle\lambda, Y_\alpha(x)\rangle\int_0^t e^{\widetilde{S}_s^x} \, dW_s^\alpha.
\end{aligned}
$$

Observe that $e^{-\widetilde{S}_s^x}(-\Delta\widetilde{S}_s^x(\omega) + \langle\nabla\widetilde{S}_s^x(\omega), \nabla\widetilde{S}_s^x(\omega)\rangle + \langle\nabla\widetilde{S}_s^x(\omega), \nabla\log\delta\rangle) = \Delta e^{-\widetilde{S}_s^x} - \langle\nabla e^{-\widetilde{S}_s^x}(\omega), \nabla\log\delta\rangle$. With

$$\psi(t, x) := \delta^{-\frac{1}{2}}(x)E[\exp(-\widetilde{S}_t^x)],$$

and assuming sufficient regularity conditions to interchange the order of integration, we thus find the following diffusion equation:

Proposition 3.7.

(3.33)
$$\frac{\partial}{\partial t}\psi = \left(\tfrac{1}{2}\Delta - \tfrac{1}{2}\delta^{\frac{1}{2}}\Delta\delta^{-\frac{1}{2}} + \tfrac{1}{2}\langle\lambda, \mathbb{I}_x^{-1}(\lambda)\rangle\right)\psi.$$

As with (3.26) and (3.25) this should be compared with [11, Thm. 4.5]. One could repeat the comments made after [11, Thm. 4.5]: The first term on the right hand side of (3.33) corresponds to the classical kinetic energy while the third corresponds to the classical potential in (3.20), and the second term ("an extra measure factor") has no trace in the classical picture. However, there is a sign difference in the exponent of this measure factor. We will come back to this in the next section and attribute it to be due to a time reversal in the construction of the projected stochastic action. We want to emphasize that the potential term $\tfrac{1}{2}\langle\lambda, \mathbb{I}_x^{-1}(\lambda)\rangle$ has purely stochastic origins namely it appears by invoking the Ito formula, and yet it coincides exactly with the classical potential that is obtained via Hamiltonian reduction.

Nelson [30] uses the stochastic Hamilton-Jacobi equation of Guerra-Morato [12] coupled with the Fokker-Planck equation to obtain an equation of the form $i\frac{\partial}{\partial t}\psi = H\psi$. While the ingredients which go into (3.33) are 'very similar' we could not produce the factor i in this setting.

3.E. **Eigenfunctions of Δ^0 and the projected stochastic action.** There were several loose ends in the previous sections: We have introduced the approaches of Nelson [30] and Lazaro-Cami and Ortega [27] to the stochastic Hamilton-Jacobi equation but not said anything about their relation. Concerning equation (3.33) one may wonder how properties of the Lagrange submanifolds L_f will relate to properties of solutions ψ, and if stationary solutions $(\tfrac{1}{2}\Delta - \tfrac{1}{2}\delta^{-\frac{1}{2}}\Delta\delta^{\frac{1}{2}} + \tfrac{1}{2}\langle\lambda, \mathbb{I}_x^{-1}(\lambda)\rangle)\psi = \gamma\psi$, for $\gamma \in \mathbb{R}$, can be characterized in terms of L_f. We can neither answer these questions in general, nor give a thourogh examination of the relations between the different approaches to the stochastic Hamilton-Jacobi equation. To obtain some partial answers we assume that $\mathcal{O} \subset \mathfrak{g}^*$ is the 0-orbit, that is

(3.34)
$$\lambda = 0,$$

from now on.

Since we can identify B and $\mathbb{R}^l$ we may determine the Hamiltonian diffusion $\Gamma^{\mathcal{O}} = (X_t, U_t) \in T^*B = \mathbb{R}^{2l}$ given by (3.31) by the Ito equation

$$d\begin{pmatrix}X_t^x \\ U_t^u\end{pmatrix} = \begin{pmatrix}\tfrac{1}{2}\nabla\log\delta(X_t^x)dt + dB \\ 0\end{pmatrix}, \qquad \Gamma_0^{\mathcal{O}} = (X_0^x, U_0^u) = (x, u) \text{ a.s.}$$

where B is Brownian motion in $\mathbb{R}^l$.

Let $\xi(x)$ be the explosion time of a solution X^x starting at x a.s. In [36, Chapter V] it is shown that the following holds: $U_t(\omega) := \{x \in B : \xi(x, \omega) > t\} \subset B$ is an open subset and $\mathrm{fl}_t(\omega) : U_t(\omega) \to B$, $x \mapsto X_t^x(\omega)$ is a diffeomorphism onto its image. Hence, by the inverse function theorem, for given x, ω and (small) t there exists

a unique point $\hat{x}(x,t,\omega)$ such that $X_t^{\hat{x}(x,t,\omega)}(\omega) = x$. Let us assume that for each x there is an ω-independent number

$$T_x > 0 \text{ with } T_x < \xi(x) \text{ a.s.}$$

such that $\hat{x}(x,t,\omega)$ exists for all $t \leq T_x$. (If not one can use a stopping time but then the time-reversed process below needs also to be defined in terms of a stopping time, and this does not seem to be standard.)

With assumption (3.34) the (projected) stochastic action of [27, Def. 3.2] associated to the smooth function f and Lagrange submanifold $L_f = \{(x, \nabla f(x)) : x \in B\} \subset TB = T^*B$ becomes

$$(3.35) \qquad \widetilde{S}_t^x(\omega) = f(\hat{x}(x,t,\omega)) + \int_0^t \big(i_{X_{h^0}}\theta - h^0\big)((\Gamma^{\mathcal{O}})_s^z(\omega))\, ds + \sum \int_0^t \big(i_{X_{h^i}}\theta - h^i\big)((\Gamma^{\mathcal{O}})_s^z(\omega))\, dB_s^i$$

where $z = z(x,t,\omega) = (\hat{x}(x,t,\omega), \nabla f(\hat{x}(x,t,\omega)))$ and θ is the Liouville one-form. Notice that h^0 and h^i are momentum functions whence $i_{X_{h^0}}\theta - h^0 = i_{X_{h^i}}\theta - h^i = 0$.

We shall denote the time reversal of X^x on the interval $[0, T_x]$ by $\hat{X}^x$. This is defined as

$$\hat{X}_t^x := X_{T_x-t}^{\hat{x}(x,T_x,\cdot)}, \quad t \in [0, T_x].$$

By [13, 36] $\hat{X}^x$ is again a diffusion and satisfies the Ito equation

$$d\hat{X}_t = -\tfrac{1}{2}\nabla \log \delta(\hat{X}_t^x)dt + dB_t.$$

Note that $\hat{X}_0^x(\omega) = X_{T_x}^{\hat{x}(x,T_x,\omega)}(\omega) = x$, $\hat{X}_{T_x}^x(\omega) = \hat{x}(x,T_x,\omega)$ and $\hat{X}_t^x(\omega) = \hat{x}(x,t,\omega)$ for $t \in [0, T_x]$. Thus we rewrite (3.35) and use the Ito formula to obtain

$$(3.36) \qquad \widetilde{S}_t^x = f(\hat{X}_t^x) = f(x) + \int_0^t \tfrac{1}{2}\big(\Delta f - \langle \nabla f, \nabla \log \delta \rangle\big)(\hat{X}_s^x)\, ds + \int_0^t \frac{\partial f}{\partial x^i}(\hat{X}_s^x)\, dB_s^i.$$

Thus $E[\widetilde{S}_t^x]$ satisfies the Kolmogorov backward equation $\frac{\partial}{\partial t}E[\widetilde{S}_t^x] = (\tfrac{1}{2}\Delta - \tfrac{1}{2}\nabla \log \delta)E[\widetilde{S}_t^x]$ and $\psi(t,x) := \delta^{-\frac{1}{2}}(x)E[\widetilde{S}_t^x]$ satisfies

$$(3.37) \qquad \frac{\partial}{\partial t}\psi = \big(\tfrac{1}{2}\Delta - \delta^{-\frac{1}{2}}\Delta\delta^{\frac{1}{2}}\big)\psi$$

which is, of course, (3.33) with $\lambda = 0$.

If f is an eigenfunction of $\Delta - \nabla \log \delta$ with $(\Delta - \nabla \log \delta)f = \gamma f$ then it follows that

$$\psi(t,x) = \delta^{-\frac{1}{2}}(x)E[\widetilde{S}_t^x] = e^{\frac{1}{2}\gamma t}\delta^{-\frac{1}{2}}(x)f(x)$$

is a stationary solution of (3.37).

To change the sign in the exponent of the measure term in (3.37) we alter the definition of the projected stochastic action (3.35): With L_f as above we define

$$(3.38) \quad S^+(x,t,\omega) := f(\tau \circ (\Gamma^{\mathcal{O}})_t^{(x,\nabla f(x))}(\omega))$$

$$+ \int_0^t \big(i_{X_{h^0}}\theta - h^0\big)((\Gamma^{\mathcal{O}})_s^{(x,\nabla f(x))}(\omega))\, ds + \sum \int_0^t \big(i_{X_{h^i}}\theta - h^i\big)((\Gamma^{\mathcal{O}})_s^{(x,\nabla f(x))}(\omega))\, dB_s^i.$$

This would be the *time-forward projected stochastic action*. With the same reasoning as above it follows that

$$(3.39) \qquad E[S^+(x,t)] = E[f \circ X_t^x] = f(x) + E\left[\int_0^t \big(\tfrac{1}{2}\Delta f + \tfrac{1}{2}\langle \nabla f, \nabla \log \delta \rangle\big)(X_s^x)\, ds\right].$$

Proposition 3.8. *The function*

$$\psi^+(x,t) := \sqrt{\delta(x)}E[S^+(x,t)]$$

satisfies $\psi^+(x,0) = \sqrt{\delta(x)}f(x)$ *and*

$$(3.40) \qquad \frac{\partial}{\partial t}\psi^+ = \tfrac{1}{2}\big(\Delta - \delta^{-\frac{1}{2}}\Delta\delta^{\frac{1}{2}}\big)\psi^+.$$

Furthermore, if f is an eigenfunction of $\Delta^0 = \Delta + \nabla \log \delta$ with eigenvalue γ then ψ^+ is a stationary solution of (3.40), i.e.

$$\psi^+(t,x) = e^{\frac{1}{2}\gamma t}\sqrt{\delta(x)}f(x).$$

Proof. Follows directly from the Kolmogorov backward equation applied to (3.39). $\qquad\square$

For $f = 1$ and $\gamma = 0$ we recover (3.26).[7]

4. REDUCTION OF BROWNIAN MOTION AND QUANTUM CALOGERO-MOSER MODELS

4.A. **The Cartan decomposition.** Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and Killing form B. Consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ associated to the Cartan involution θ, and let $G = K.\exp(\mathfrak{p}) \cong K \times \mathfrak{p}$, $g = k \exp x \leftrightarrow (k, x)$ be the corresponding decomposition of the group. Thus:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, and put $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ and $M = Z_K(\mathfrak{a})$.

Let Σ be the set of restricted roots associated to the pair $(\mathfrak{g}, \mathfrak{a})$ and $\Sigma_+ \subset \Sigma$ a choice of positive roots. Then the associated root space decomposition is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \text{ where } \mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}.$$

We equip $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with the Ad-invariant direct sum inner product $\langle ., . \rangle = -B(., \theta.) = -B|(\mathfrak{k} \times \mathfrak{k}) + B|(\mathfrak{p} \times \mathfrak{p})$ and denote its restriction to $\mathfrak{k}$ or $\mathfrak{p}$ by $\langle ., . \rangle$ again.

4.B. **The classical rational Calogero-Moser system associated to a Cartan decomposition.** Let $Q \subset \mathfrak{p}$ denote the set of points which are regular with respect to the K-action. Thus all isotropy subgroups K_q are conjugate to M within K and the quotient $Q/K = C$ is an open Weyl chamber. The Lagrangian L is the kinetic energy function

$$L : TQ = Q \times \mathfrak{p} \longrightarrow \mathbb{R}, \quad (q, v) \longmapsto \tfrac{1}{2}\langle v, v \rangle = \tfrac{1}{2}\mu(v, v)$$

which is clearly K-invariant with respect to the tangent lifted K-action. Identifying $TQ = T^*Q$ via the metric we can equip TQ with the standard symplectic form Ω and view L as a Hamiltonian function H. Associated to the K-action there is now a momentum map

$$J : TQ \longrightarrow \mathfrak{k}^* =_{\langle .,. \rangle} \mathfrak{k}, \quad (q, v) \longmapsto \mathrm{ad}(q).v = [q, v].$$

Consider a (co-)adjoint orbit $\mathcal{O} = \mathrm{Ad}(K).\lambda \subset \mathfrak{k}$. Then the Hamiltonian reduction of (TQ, Ω, H) at the level $\mathcal{O}$ yields a reduced phase space of the form

(4.41) $$TQ /\!/_{\mathcal{O}} K \cong TC \times \mathcal{O} /\!/_0 M \text{ where } \mathcal{O} /\!/_0 M = (\mathcal{O} \cap \mathfrak{m}^\perp)/M.$$

This isomorphism can be realized in terms of the mechanical connection defined in (3.16). The space $(\mathcal{O} \cap \mathfrak{m}^\perp)/M$ is, generally, not a manifold but rather a stratified space. Accordingly $TQ /\!/_{\mathcal{O}} K$ is a stratified space, and each stratum $TC \times$ (stratum) is equipped with a product symplectic form such that the first factor is canonical while the second is inherited from the KKS-form on $\mathcal{O}$. The reduced Hamiltonian becomes, in this picture, the Calogero-Moser Hamiltonian function

$$H_{\mathrm{CM}}(x, v_0, [\lambda]) = \tfrac{1}{2}\langle v_0, v_0 \rangle + \tfrac{1}{2} \sum_{\alpha \in \Sigma_+} \frac{\langle \lambda, \lambda \rangle}{\alpha(x)^2}$$

Note that the potential term $\tfrac{1}{2}\sum_{\alpha \in \Sigma_+} \frac{\langle \lambda, \lambda \rangle}{\alpha(x)^2} = \tfrac{1}{2}\langle A_x^* \lambda, A_x^* \lambda \rangle = \tfrac{1}{2}\langle \lambda, \mathbb{I}_x^{-1}(\lambda) \rangle$ can be seen as an M-invariant function on $C \times (\mathcal{O} \cap \mathfrak{m}^\perp)$. See [1, 15, 9].

---

[7]Stationary solutions of (3.40) correspond to stationary solutions of the corresponding Schrödinger equation $i\frac{\partial}{\partial t}\phi = -\frac{1}{2}(\Delta - \delta^{-\frac{1}{2}}\Delta\delta^{\frac{1}{2}})\phi$.

4.C. **Stochastic reduction and Quantum CM systems.** As in Section 3.E we set $\mathcal{O} = 0$ in (4.41) and hence $TQ /\!/_{\mathcal{O}} K = TQ /\!/_0 K = T(Q/K) = TC \cong \mathbb{R}^{2l}$. Let $f : C \to \mathbb{R}$ be smooth and consider the forward projected stochastic action

$$S^+(x, t) = f \circ X_t^x$$

from (3.38) associated to the Lagrange submanifold $L_f = \{(x, \nabla f(x))\} \subset TC$. As shown in Section 3.E, if f is an eigenfunction of $\Delta^0 = \Delta + \nabla \log \delta$ it follows that

$$\psi^+(t, x) = \sqrt{\delta(x)} E[S^+(x, t)] = e^{\frac{1}{2}\gamma t} \sqrt{\delta(x)} f(x)$$

is a stationary solution to $\frac{\partial}{\partial t} - \frac{1}{2}(\Delta - \delta^{-\frac{1}{2}}\Delta\delta^{\frac{1}{2}}) = 0$. Such functions are provided by the zonal spherical functions. See [33, Appendix C]. According to [33, Theorem 5.1] we have

$$\tfrac{1}{2}\delta^{-\frac{1}{2}}\Delta\delta^{\frac{1}{2}} = \sum_{\alpha \in \Sigma_+} \tfrac{m_\alpha(2m_\alpha - 2)|\alpha|^2}{8\alpha(x)^2}.$$

Thus

$$\psi(t, x) = \psi^+(it, x) = e^{i\frac{1}{2}\gamma t} \sqrt{\delta(x)} f(x)$$

is a stationary solution to the Schrödinger equation

$$(4.42) \qquad i\tfrac{\partial}{\partial t}\psi = \left(-\tfrac{1}{2}\Delta + \sum_{\alpha \in \Sigma_+} \tfrac{m_\alpha(2m_\alpha - 2)|\alpha|^2}{8\alpha(x)^2} \right) \psi$$

where $m_\alpha = \dim \mathfrak{g}_\alpha$ is the multiplicity of the root. Equation (4.42) is the rational quantum Calogero-Moser Schrödinger equation related to root systems of semi-simple Lie algebras as described by Olshanetsky and Perelomov [32, 33].

It will be a matter of future investigation to consider reduction at a non-zero orbit $\mathcal{O}$ and include a spin dependent potential in (4.42) as in (3.33).

5. APPENDIX: PROOF OF THEOREM 1.3

Theorem 1.3 is a result of [19]. However, as this paper is presently not published, we follow the referees' suggestion and include a (slightly shortened) proof for sake of completeness.

Proof of Theorem 1.3. Let us begin by noting that $g\Gamma^{x,W} = \Gamma^{gx,\rho(g)W}$. Indeed,

$$\delta(g\Gamma^{x,W}) = g\mathcal{S}(Y, \Gamma^{x,W})\delta Y = \mathcal{S}(\rho(g)Y, g\Gamma^{x,W})\delta(\rho(g)Y)$$

whence $\tilde{\Gamma} := g\Gamma^{x,W}$ satisfies $\tilde{\Gamma}_0 = gx$ a.s. and $\delta\tilde{\Gamma} = \mathcal{S}(\rho(g)Y, \tilde{\Gamma})\delta(\rho(g)Y)$. By existence and uniqueness of solutions the claim follows. In particular, we have $\pi \circ \Gamma^{x,W} = \pi \circ \Gamma^{gx,\rho(g)W}$.

Claim:

$$(5.43) \qquad\qquad\qquad\qquad P_{gx} = g_* P_x$$

where G acts on $W(Q)$ as $g : w \mapsto (t \mapsto gw(t))$. To see this let $S \subset W(Q)$ be a Borel cylinder set. This means that there are $l \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_l \in \mathbb{R}_+$, and a Borel set $A \subset \Pi^l \dot{Q}$ such that $S = \mathrm{ev}(t_1, \ldots, t_l)^{-1}(A)$, where $\mathrm{ev}(t_1, \ldots, t_l) : W(Q) \to \Pi^l \dot{Q}$, $w \mapsto (w(t_i))_{i=1}^l$. From the identity $(\Gamma^{x,\rho(g)W})\check{\,}_* P = (\Gamma^{x,W})\check{\,}_* P$ we find

$$P_{gx}(S) = (\Gamma^{gx,\rho(g)W})\check{\,}_* P(S) = P\{\omega : (\Gamma_{t_i}^{gx,\rho(g)W}(\omega))_{i=1}^l \in A\}$$
$$= P_x(\mathrm{ev}(t_1, \ldots, t_l)^{-1}(g^{-1}A)) = P_x(g^{-1}S).$$

Consider the push forward map $\pi_* : W(Q) \to W(Q/G)$, $w \mapsto \pi \circ w$. It is straightforward to see that $\mathcal{B}(W(Q/G)) = \pi_*\mathcal{B}(W(Q))$. For $S_0 = \pi_*(S) \in \mathcal{B}(W(Q/G))$ we may write the law $\left(P_{[x]}\right)_{[x] \in \dot{Q}/G}$ of $\pi \circ \Gamma$ as

$$P_{[x]}(S_0) = (\pi \circ \Gamma^{gx,\rho(g)W})\check{\,}_* P(S_0) = P_{gx}(\pi_*^{-1}(S_0)).$$

By (5.43) this does not depend on $g \in G$.

Now, since $P_{[x]}$ is the push-forward of P_x via p, we can use the strong Markov property of $(P_x)_x$ to conclude that $(P_{[x]})_{[x] \in \dot{Q}/G}$ satisfies the strong Markov property.

To show that $\sum X_i X_i f \in C^\infty(Q)^G$ for all $f \in C^\infty(Q)^G$ consider the standard basis $\{e_0, e_1, \ldots, e_k\}$ of $\mathbb{R} \times \mathbb{R}^k$. For $j = 1, \ldots, k$ we find

$$g \cdot X_j(x) = g \cdot \mathcal{S}(x, y, e_j) = \mathcal{S}(gx, \rho(g)y, \rho(g)e_j) = \sum_k g_{kj} X_k(gx),$$

where $g_{kj} := \langle e_k, \rho(g)e_j \rangle$ is independent of $x \in Q$. Since $\sum_j g_{ij} g_{kj} = \delta_{ik}$,

$$X_i(gx) = \sum_{j,k} g_{ij} g_{kj} X_k(gx) = \sum_j g_{ij} g \cdot X_j(x).$$

Thus $\left(df(X_i)\right)(gx) = \sum_j g_{ij}\left(df(X_j)\right)(x)$ for $f \in C^\infty(Q)^G$ and also

$$d\left(df(X_i)\right)(gx) \circ T_x g = d\left(\sum_j g_{ij} df(X_j)\right)(x) = \sum_j g_{ij} d\left(df(X_j)\right)(x).$$

This implies that

$$\sum_i \left(X_i X_i f\right)(gx) = \sum_i \left\langle d\left(df(X_i)\right)(gx), X_i(gx) \right\rangle$$

$$= \sum_{i,j,k} \left\langle g_{ij} d\left(df(X_j)\right)(x) \circ (T_x g)^{-1}, g_{ik}(T_x g) \cdot X_k(x) \right\rangle = \sum_i \left(X_i X_i f\right)(x).$$

Similarly, it is also easy to see that X_0 is G-invariant. Thus the generator $A = X_0 + \frac{1}{2} \sum X_i X_i$ acts on $C^\infty(Q)^G$, whence it induces a projected operator A_0 characterized by $A \circ \pi^* = \pi^* \circ A_0$.

Finally, to see that A_0 is the generator of $\pi \circ \Gamma$ we need to show that, for all $t \in \mathbb{R}_+$, $[x] \in Q/G$, and $f \in C^\infty(Q/G)_0$, the $\mathbb{R}$-valued process

$$M_t^f : \mathcal{W}(Q/G) \longrightarrow \mathbb{R},$$

$$M_t^f(w) := f(w(t)) - f(w(0)) - \int_0^t (A_0 f)(w(s)) \, ds$$

is a $P_{[x]}$-martingale on $(W(Q/G), \mathcal{B}(W(Q/G))$ for the filtration $(\mathcal{B}_t(W(Q/G)))_t$. See [20, Def. IV.5.3]. This means that for all $t \geq 0$, $s \in [0, t]$, and $A \in \mathcal{B}_s(W(Q/G))$ we should check that

$$\int_A E^{P_{[x]}}\left[M_t^f \Big| \mathcal{B}_s(W(Q/G))\right](w) \, P_{[x]}(dw) = \int_A M_s^f(w) \, P_{[x]}(dw);$$

see [38, Chapter V]. Indeed,

$$\int_A E^{P_{[x]}}\left[M_t^f \Big| \mathcal{B}_s(W(Q/G))\right](w) \, P_{[x]}(dw) = \int_A M_t^f(w) \, P_{[x]}(dw) = \int_{p^{-1}A} (p^* M_t^f)(u) \, P_x(du)$$

$$= \int_{p^{-1}A} (p^* M_s^f)(u) \, P_x(du) = \int_A M_s^f(w) \, P_{[x]}(dw).$$

$\square$

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